University of California, Berkeley
Physics H7A Fall 1998 (*Strovink*)

SOLUTION TO PROBLEM SET 1

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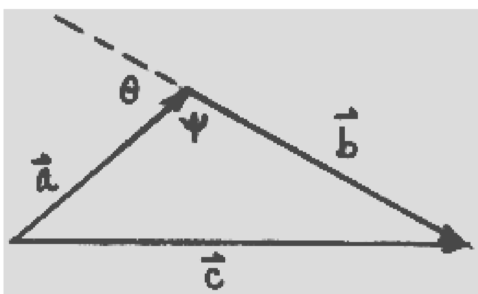
1. You may remember the law of cosines from trigonometry. It will be useful for several parts of this problem, so we will state it here. If the lengths of the sides of a triangle are a, b, and c, and the angle opposite the side c is ψ, then

$$c^2 = a^2 + b^2 - 2ab\cos\psi$$

(a.) When two vectors add up to a third vector, the three vectors form a triangle. If the angle between $\mathbf{a}$ and $\mathbf{b}$ is θ, then the angle opposite the side formed by $\mathbf{c}$ is $180° - \theta$.



The law of cosines then tells us that

$$|\mathbf{c}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2|\mathbf{a}||\mathbf{b}|\cos(180° - \theta)$$

From trigonometry, remember that

$$\cos(180° - \theta) = -\cos\theta$$

which gives

$$|\mathbf{c}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 + 2|\mathbf{a}||\mathbf{b}|\cos\theta$$

We know that $|\mathbf{a}| + |\mathbf{b}| = |\mathbf{c}|$. Squaring this equation, we get

$$|\mathbf{c}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 + 2|\mathbf{a}||\mathbf{b}|$$

If we compare this with the equation above, we can see that $\cos\theta$ has to be equal to one. This only happens when $\theta = 0°$. What this means is that the two vectors are parallel to each other, and they point in the same direction. If the angle between them were $180°$, then they would be parallel but point in opposite directions.

(b.) This part is simple. Just subtract the vector $\mathbf{a}$ from both sides to see that $\mathbf{b} = -\mathbf{b}$. The only way that this can happen is if $\mathbf{b} = \mathbf{0}$, the zero vector.

(c.) This part can also be done by the law of cosines. Like part (a.), we have the following two equations

$$|\mathbf{c}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 + 2|\mathbf{a}||\mathbf{b}|\cos\theta$$

This is just the law of cosines again, where θ is the angle between $|\mathbf{a}|$ and $\mathbf{b}|$. The problem states that

$$|\mathbf{c}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2$$

Comparing this with the equation above, we find that $\cos\theta = 0$. This happens at $\theta = \pm 90°$. This means that the vectors must be perpendicular to each other.

(d.) Yet again, we can use the law of cosines. If the angle between $\mathbf{a}$ and $\mathbf{b}$ is θ, then the angle between $\mathbf{a}$ and $-\mathbf{b}$ is $180° - \theta$. The lengths of the sum and difference are

$$|\mathbf{a} + \mathbf{b}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 + 2|\mathbf{a}||\mathbf{b}|\cos\theta$$
$$|\mathbf{a} - \mathbf{b}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2|\mathbf{a}||\mathbf{b}|\cos\theta$$

For these to be equal, we need $\cos\theta = 0$, which happens when $\theta = \pm 90°$. Again, this means that the vectors are perpendicular.

(e.) Guess what? Yup, law of cosines. We know that $|\mathbf{a}| = |\mathbf{b}| = |\mathbf{a} + \mathbf{b}|$. Adding $\mathbf{a}$ to $\mathbf{b}$ is going to look like two vectors stuck together to form two sides of a triangle. If the angle between the vectors is θ, the law of cosines gives

$$|\mathbf{a} + \mathbf{b}|^2 = 2|\mathbf{a}|^2 + 2|\mathbf{a}|^2\cos\theta$$

where we have used the fact that $\mathbf{a}$ and $\mathbf{b}$ have the same length. We also know that $|\mathbf{a}+\mathbf{b}| = |\mathbf{a}|$. Using this we get

$$|\mathbf{a}|^2 = 2|\mathbf{a}|^2 + 2|\mathbf{a}|^2\cos\theta$$

Dividing by $|\mathbf{a}|^2$, we find a condition on the angle θ

$$\cos\theta = -\frac{1}{2} \quad \Rightarrow \quad \theta = 120°$$

2. K&K problem 1.2

We can use the dot product, also known as the inner product, of two vectors here. Remember that

$$\mathbf{A}\cdot\mathbf{B} = |\mathbf{A}||\mathbf{B}|\cos\theta$$

where θ is the angle between the vectors. We can use the formula for computing the dot product from the vector components

$$\mathbf{A}\cdot\mathbf{B} = A_x B_x + A_y B_y + A_z B_z$$

The vectors are given as follows: $\mathbf{A} = 3\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$ and $\mathbf{B} = -2\hat{\mathbf{i}} - 3\hat{\mathbf{j}} - \hat{\mathbf{k}}$. Multiplying, we find that $\mathbf{A}\cdot\mathbf{B} = -10$. We need the lengths of $\mathbf{A}$ and $\mathbf{B}$. Remember that $|\mathbf{A}|^2 = \mathbf{A}\cdot\mathbf{A}$. This tells us that $|\mathbf{A}|^2 = 11$ and $|\mathbf{B}|^2 = 14$. Dividing, we find that

$$\cos\theta = \frac{-10}{\sqrt{11\cdot 14}} = -0.805$$

3. Using the formulas on the problem set, we can convert the points on the surface of the sphere to Cartesian coordinates.

$$\begin{aligned} x_1 &= R\sin\theta_1\cos\phi_1 \\ y_1 &= R\sin\theta_1\sin\phi_1 \\ z_1 &= R\cos\theta_1 \\ x_2 &= R\sin\theta_2\cos\phi_2 \\ y_2 &= R\sin\theta_2\sin\phi_2 \\ z_2 &= R\cos\theta_2 \end{aligned}$$

As in problem 2, we need to know the length of these vectors in order to calculate the angle between them from the dot product. It is fairly obvious that the lengths of the vectors are just R, because that is the radius of the sphere; we will show this explicitly.

$$\begin{aligned} |(R,\theta,\phi)|^2 = R^2(&\sin^2\theta\cos^2\phi \\ &+ \sin^2\theta\sin^2\phi + \cos^2\theta) \end{aligned}$$

Using the fact that $\sin^2\phi + \cos^2\phi = 1$, we get

$$|(R,\theta,\phi)|^2 = R^2\left(\sin^2\theta + \cos^2\theta\right)$$

We can just repeat the previous step for θ now and get

$$|(R,\theta,\phi)| = R$$

as we expected in the first place. Now we calculate the dot product. Let $\mathbf{x}_1$ be the vector to the first point and $\mathbf{x}_2$ be the vector to the second point. We find that

$$\begin{aligned} \mathbf{x}_1\cdot\mathbf{x}_2 = R^2(&\sin\theta_1\sin\theta_2\cos\phi_1\cos\phi_2 \\ &+ \sin\theta_1\sin\theta_2\sin\phi_1\sin\phi_2 + \cos\theta_1\cos\theta_2) \end{aligned}$$

This can be simplified if we remember the formula for the cosine of a sum of two angles.

$$\cos(\theta\pm\phi) = \cos\theta\cos\phi \mp \sin\theta\sin\phi$$

Using this formula, we get the result

$$\begin{aligned} \mathbf{x}_1\cdot\mathbf{x}_2 = R^2(&\sin\theta_1\sin\theta_2\cos(\phi_1-\phi_2) \\ &+ \cos\theta_1\cos\theta_2) \end{aligned}$$

To get the angle we just divide by the lengths of each vector, which are both R. This gives the final result.

$$\begin{aligned} \cos\theta_{12} = &\cos\theta_1\cos\theta_2 \\ &+ \sin\theta_1\sin\theta_2\cos(\phi_1-\phi_2) \end{aligned}$$

4. This problem is an application of the results of problem 3.

(a.) A straight tunnel between Sydney and New York can be represented by the *difference* of the vectors pointing to their locations. To say it another way, the distance between the ends of two vectors is the length of the difference of the vectors. Adjusting for the fact that latitude and longitude are not quite the same as the coordinates θ and ϕ, we find the polar coordinates of the cities.

$$\begin{aligned} \mathbf{X}_{\mathrm{NY}} &= (6370\,\mathrm{km}, 49^\circ, 286^\circ) \\ \mathbf{X}_{\mathrm{Sydney}} &= (6370\,\mathrm{km}, 124^\circ, 151^\circ) \end{aligned}$$

Converting to Cartesian coordinates (x, y, z) using the formulas from problem 3 we get

$$\begin{aligned} \mathbf{X}_{\mathrm{NY}} &= (1325\,\mathrm{km}, -4621\,\mathrm{km}, 4179\,\mathrm{km}) \\ \mathbf{X}_{\mathrm{Sydney}} &= (-4619\,\mathrm{km}, 2560\,\mathrm{km}, -3562\,\mathrm{km}) \end{aligned}$$

The distance between New York and Sydney through the earth is just $|\mathbf{X}_{\text{NY}} - \mathbf{X}_{\text{Sydney}}|$. The result of the calculation is

$$\text{Distance} = 12,117\,\text{km}$$

(b.) Using the result from problem 3 to calculate the angle between Sydney and New York, we find that $\cos\theta_{12} = -0.809$, thus $\theta_{12} = 144.0°$. To calculate the distance along the earth's surface we need to express this angle in radians. The conversion formula is

$$\theta(\text{radians}) = \frac{\pi}{180°}\theta(\text{degrees})$$

Thus $\theta_{12} = 2.513$ radians. Multiplying this by the radius of the earth, we get the "great circle" distance between New York and Sydney:

$$\text{Distance} = 16,010\,\text{km}$$

5. K&K problem 1.6

This question asks you to prove the law of sines using the cross product. Let A, B, and C be the lengths of the vectors making the three sides of the triangle. Let a, b and c be the angle opposite each of those sides. The law of sines states that

$$\frac{\sin a}{A} = \frac{\sin b}{B} = \frac{\sin c}{C}$$

Remember that the *length* of the cross product of two vectors is equal to the *area* of the parallelogram defined by them. Remember also that the the length of the cross product is equal to the product of the lengths times the sine of the angle between them: $|\mathbf{A}\times\mathbf{B}| = |\mathbf{A}||\mathbf{B}|\sin\theta$. We have three vectors to play with in this problem, and using the cross product we can compute the area of the triangle from any two of them. We find that

$$\text{Area} = AB\sin c = BC\sin a = AC\sin b$$

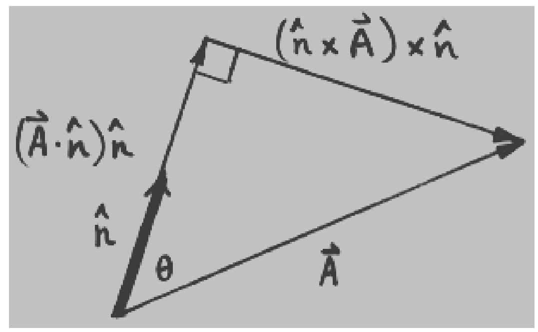
We just divide the whole thing by ABC and we recover the law of sines.

6. K&K problem 1.11

Let $\mathbf{A}$ be an arbitrary vector and let $\hat{\mathbf{n}}$ be a unit vector in some fixed direction. Show that

$$\mathbf{A} = (\mathbf{A}\cdot\hat{\mathbf{n}})\,\hat{\mathbf{n}} + (\hat{\mathbf{n}}\times\mathbf{A})\times\hat{\mathbf{n}}$$

Form a triangle from the three vectors in this equation. Let $\mathbf{B} = (\mathbf{A}\cdot\hat{\mathbf{n}})\,\hat{\mathbf{n}}$ and let $\mathbf{C} = (\hat{\mathbf{n}}\times\mathbf{A})\times\hat{\mathbf{n}}$. Let the angle between $\mathbf{A}$ and $\hat{\mathbf{n}}$ be θ. What this formula does is to break up the vector $\mathbf{A}$ into a piece parallel to $\hat{\mathbf{n}}$ and a piece perpendicular to $\hat{\mathbf{n}}$. $\mathbf{B}$ gives the parallel piece. Its length is just $|\mathbf{B}| = |\mathbf{A}|\cos\theta$. The length of the perpendicular piece must then be $|\mathbf{A}|\sin\theta$.



Examining the vector $\mathbf{C}$, we see that inside the parentheses is a vector whose length is $|\mathbf{A}|\sin\theta$ and is perpendicular to $\hat{\mathbf{n}}$. This vector is then crossed into $\hat{\mathbf{n}}$. Since it is perpendicular to $\hat{\mathbf{n}}$, the length of the final vector is $|\mathbf{A}|\sin\theta$, which is what we want. Now we are just concerned with the direction. The first cross product is perpendicular to the plane containing $\hat{\mathbf{n}}$ and $\mathbf{A}$. The second cross product is perpendicular to the first, thus it is coplanar with $\hat{\mathbf{n}}$ and $\mathbf{A}$. It is also perpendicular to $\hat{\mathbf{n}}$. Thus it represents the component of $\mathbf{A}$ that is perpendicular to $\hat{\mathbf{n}}$. Be careful about the sign here.

A useful vector identity that you will be seeing again is the so-called "BAC-CAB" rule. It is an identity for the triple cross product.

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B})$$

Its fairly obvious why this is called the BAC-CAB rule. Using this rule, we see that

$$\hat{\mathbf{n}}\times(\hat{\mathbf{n}}\times\mathbf{A}) = \hat{\mathbf{n}}(\hat{\mathbf{n}}\cdot\mathbf{A}) - \mathbf{A}(\hat{\mathbf{n}}\cdot\hat{\mathbf{n}})$$

This immediately gives

$$\mathbf{A} = (\mathbf{A}\cdot\hat{\mathbf{n}})\,\hat{\mathbf{n}} + (\hat{\mathbf{n}}\times\mathbf{A})\times\hat{\mathbf{n}}$$

Of course we haven't derived the BAC-CAB rule here. It's a mess.

7. The idea in all of the parts of this problem is that the plane must oppose any perpendicular wind speed to maintain its straight path. If the wind is blowing with a speed v perpendicular to the path, the plane's airspeed must be $-v$ perpendicular to the path. The airspeed is $\mathbf{u}$, the wind speed relative to the ground is $\mathbf{w}$, and the ground speed is $\mathbf{v} = \mathbf{u} + \mathbf{w}$. $|\mathbf{u}| = U_0$. Let the total distance traveled be D.

(a.) No wind, $\mathbf{w} = \mathbf{0}$ so $\mathbf{u} = \mathbf{v}$. $T = D/U_0$.

(b.) Wind of speed W_0 blowing parallel to the path. When the wind is going with the plane, $v = W_0 + U_0$, when it opposes the plane, the ground speed is $v = U_0 - W_0$. The time for the first leg is $T_1 = D/2(U_0 + W_0)$. The time for the second leg is $T_2 = D/2(U_0 - W_0)$. The total time is the sum

$$T = \frac{D}{2}\left(\frac{1}{U_0 + W_0} + \frac{1}{U_0 - W_0}\right)$$

This can be simplified, and we get the final answer, which agrees with part (a.) when $W_0 = 0$.

$$T = \frac{DU_0}{U_0^2 - W_0^2}$$

(c.) Wind of speed W_0 blowing perpendicular to the path. This part is a little harder. The plane will not be pointed straight along the path because it has to oppose the wind trying to blow it off course. The airspeed of the plane in the perpendicular direction will be W_0, and we know what the total airspeed is, so we can calculate the airspeed along the path.

$$U_0^2 = U_\perp^2 + U_\parallel^2 \quad\Rightarrow\quad U_\parallel = \sqrt{U_0^2 - W_0^2}$$

The wind has no component along the path of motion, so the airspeed in the parallel direction is the same as the ground speed in the parallel direction. The ground speed is furthermore the same on both legs of the trip. The final answer again agrees with part (a.) when $W_0 = 0$

$$T = \frac{D}{\sqrt{U_0^2 - W_0^2}}$$

(d.) Wind of speed W_0 blowing at an angle θ to the direction of travel. The plane again needs to cancel the component of the wind blowing in the perpendicular direction. The perpendicular component of the wind speed is $W_{0\perp} = W_0 \sin\theta$. As in part (c.) the airspeed in the parallel direction can be computed

$$U_0^2 = U_\perp^2 + U_\parallel^2 \quad\Rightarrow\quad U_\parallel = \sqrt{U_0^2 - W_0^2 \sin^2\theta}$$

In this case, the wind has a component along the direction of travel. This parallel component is $W_{0\parallel} = W_0 \cos\theta$. On one leg of the trip, this adds to the ground velocity. On the other leg, it subtracts. This gives us the following formula:

$$T = \frac{D}{2}\left(\frac{1}{\sqrt{U_0^2 - W_0^2 \sin^2\theta} + W_0\cos\theta} + \frac{1}{\sqrt{U_0^2 - W_0^2 \sin^2\theta} - W_0\cos\theta}\right)$$

This can be simplified considerably:

$$T = \frac{D\sqrt{U_0^2 - W_0^2 \sin^2\theta}}{U_0^2 - W_0^2}$$

If you look at this carefully, you will realize that it reduces to the correct answer for parts (a.), (b.), and (c.) with the proper values for W_0 and θ. If $\theta = 90°$, the wind blows perpendicular to the path and we get the result from part (c.). If $\theta = 0°$, the wind blows parallel to the direction of travel and we recover the result from part (b.).

(e.) (f.) This part requires some calculus. We need to do a minimization of a function. What this part asks is to study the travel time as a function of wind speed for an arbitrary angle θ. We need to consider the result from part (d.) as a function of the wind speed:

$$T(W_0) = \frac{D\sqrt{U_0^2 - W_0^2 \sin^2\theta}}{U_0^2 - W_0^2}$$

For now we are going to ignore the fact that it also depends on U_0 and θ. Remember that functions have maxima and minima at places where the derivative vanishes, so we need to take the derivative of T with respect to W_0:

$$\frac{d}{dW_0}T(W_0) = D\Bigg(\frac{2W_0\sqrt{U_0^2 - W_0^2\sin^2\theta}}{(U_0^2 - W_0^2)^2} - \frac{W_0\sin^2\theta}{\sqrt{U_0^2 - W_0^2\sin^2\theta}(U_0^2 - W_0^2)}\Bigg)$$

The derivative is clearly zero when $W_0 = 0$. In this case the travel time $T = D/U_0$ as in part (a.). There is another case we have to worry about though. We divide out what we can to get an equation for another value where the derivative vanishes

$$U_0^2\sin^2\theta - W_0^2\sin^2\theta = 2U_0^2 - 2W_0^2\sin^2\theta$$

This gives us the other point where the derivative is zero

$$W_0^2 = U_0^2\frac{2-\sin^2\theta}{\sin^2\theta}$$

Notice that this point always occurs when the wind speed is greater than the air speed. No progress can be made against the wind if this is the case, so the trip cannot occur. The final possibility to consider is the case where the wind speed is the same as the air speed. Looking at the formula, the time taken is infinite. The only possibility is that the minimum is at $W_0 = 0$. The final piece of this problem is to observe what happens to the "time taken" when $W_0 > U_0$. For one thing, it becomes negative. In some circumstances it can even become imaginary. There is really no interpretation of this other than "ask a stupid question, get a stupid answer". The answer doesn't make sense because the question didn't make sense. The trip cannot occur when $W_0 > U_0$, so it is meaningless to ask how long it would take.

8. A particle moves along the curve $y = Ax^2$ and its x position is given by $x = Bt$.

(a.) We can just plug the x equation into the y equation to get the y position as a function of time, $y = AB^2t^2$. In vector form, the position is then

$$\mathbf{r}(t) = \hat{\mathbf{x}}Bt + \hat{\mathbf{y}}AB^2t^2$$

(b.) The vector velocity is obtained from the vector position by differentiating with respect to t

$$\mathbf{v}(t) = \frac{d}{dt}\mathbf{r}(t) = \hat{\mathbf{x}}B + \hat{\mathbf{y}}2AB^2t$$

(c.) The vector acceleration is obtained from the vector velocity by again differentiating with respect to t

$$\mathbf{a}(t) = \frac{d}{dt}\mathbf{v}(t) = \hat{\mathbf{y}}2AB^2$$

(d.) The scalar speed is just the length of the velocity vector. Remember that $|\mathbf{A}| = \sqrt{\mathbf{A}\cdot\mathbf{A}}$.

$$|\mathbf{v}(t)| = \sqrt{\mathbf{v}(t)\cdot\mathbf{v}(t)} = \sqrt{B^2 + 4A^2B^4t^2}$$

(e.) The vector average velocity is the integral of the velocity vector over a time interval, divided by the time interval. In general, the (time) average of a quantity $\mathbf{A}$ is given by

$$\langle\mathbf{A}\rangle = \frac{1}{(t_2 - t_1)}\int_{t_1}^{t_2}\mathbf{A}(t)dt$$

Applying this formula, we see the integral that needs to be evaluated:

$$\begin{aligned}\langle\mathbf{v}(t_0)\rangle &= \frac{1}{t_0}\int_0^{t_0}\mathbf{v}(t)dt \\ &= \frac{1}{t_0}\int_0^{t_0}(\hat{\mathbf{x}}B + \hat{\mathbf{y}}2AB^2t)dt\end{aligned}$$

We could also use the fact that the integral of the velocity is the position to get a simpler looking formula for the average velocity

$$\langle\mathbf{v}(t_0)\rangle = \frac{1}{t_0}\left(\mathbf{r}(t_0) - \mathbf{r}(0)\right)$$

Evaluating this integral, we get an answer that is not surprising

$$\langle\mathbf{v}(t_0)\rangle = \hat{\mathbf{x}}B + \hat{\mathbf{y}}AB^2t_0$$

This is just $(\mathbf{r}(t_0) - \mathbf{r}(0))/t_0$! The average velocity is just the distance traveled divided by the time it took.

9. The idea behind this problem is to make a graph of position vs. time data and show that they fit the equation $s = a(t - t_0)^2/2$. In addition you are supposed to find t_0. The way to do this is to plot the square root of the distance vs. time, which will give a straight line graph: $\sqrt{s} = \sqrt{(a/2)}(t - t_0)$. The slope of this graph is approximately 0.168, so we can use that to extrapolate back to zero. We find that the graph reaches zero at about $t = -4.45$, so this means that $t_0 = -4.45$ to make the distance traveled equal zero at $t = -4.45$.